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Theoretical Note

On using the fixed-point property of binary mixtures to discriminate among models of recognition memory☆

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ABSTRACT

A variety of different recognition-memory models make different psychological assumptions, but similar predictions about ROC curves in old–new recognition-memory experiments. Some models assume that recognition responses are produced by a unitary process and other models assume they are a binary mixture of two qualitatively different types of responses. This note shows that despite their similar ROC predictions, the binary-mixture models make some striking predictions that the unitary models do not make. Specifically, in any experiment that includes conditions in which the mixture probability varies but the component distributions do not, the binary-mixture models predict that all response time probability density functions must intersect at the same time point (if they intersect at all). Similarly, they also all predict that if the ROC curves intersect, they must also all intersect at the same point.

1. Introduction

In old–new recognition-memory experiments, participants are presented with a list of words and then sometime later presented with a series of single words and asked to respond whether each of these single words is old or new — that is, they are asked to indicate whether each word was or was not on the studied list. New items are typically called "lures" and old items are called "targets". In the language of the YES–NO detection task, responding OLD on target trials is analogous to a hit, and responding OLD on lure trials is analogous to a false alarm. In this way, an ROC curve can be constructed for this task by plotting the probability of responding OLD on target trials on the ordinate against the probability of responding OLD on lure trials on the abscissa — that is, by plotting $P(\text{OLD}|\text{target})$ against $P(\text{OLD}|\text{lure})$. Empirical ROCs estimated from confidence judgments collected in the old–new recognition-memory task are curved, rather than linear, and when plotted in Z space, the best-fitting lines have a slope that can vary considerably across conditions, but that is typically less than 1.0 (e.g., Glanzer et al., 1999; Ratcliff et al., 1992). It is commonly assumed that a representative slope is around 0.8 (Ratcliff et al., 1992; Wixted, 2007). A variety of different signal-detection theory-based (SDT) models of performance in this task have been proposed to account for these results. Some models assume that all OLD responses are mediated by the same psychological process, whereas others assume that OLD responses are a binary-mixture of two qualitatively different types of responses. Models in the former class include the normal, unequal-variance SDT model, whereas models in the latter class include the dual-process SDT model and the mixture SDT model. Unitary and binary-mixture models make qualitatively different assumptions about the underlying psychological processes that mediate responding in the old–new recognition-memory task, but nevertheless they make similar, albeit not identical, ROC curve predictions. The similarity of these predictions has made it difficult to differentiate them empirically. This note describes some striking (untested) empirical predictions that discriminate the unitary and binary-mixture models.

2. The models

Three different SDT-based models of the old–new recognition-memory task are popular: the normal, unequal-variance SDT model (Wixted, 2007), the dual-process SDT model (Yonelinas, 1994), and the mixture SDT model (DeCarlo, 2002). The normal, unequal-variance SDT model assumes that all OLD versus NEW recognition-memory responses are based on the memory strength elicited by the presented stimulus. Specifically, the model assumes that the participant will respond OLD if the memory strength elicited by the presented stimulus is large and NEW if the memory strength is small. The model further assumes that lures and targets both generate a range of memory strengths that are each normally distributed, but on average, targets generate larger and more variable memory strengths than lures (Wixted, 2007).

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This model has three free parameters — the mean and variance of the target distribution and the criterion on memory strength for responding OLD. A model in which the target standard deviation is 1.25, given a lure standard deviation of 1.0, predicts a linear ROC in Z space with a slope of 0.8, which matches typical empirical estimates.

The normal, unequal-variance model assumes that all OLD responses depend only on a single memory-strength value, and therefore that all OLD responses are mediated by the same psychological process. In contrast, dual-process models assume that a judgment that an item is old depends on separate recollection and familiarity processes (Yonelinas, 1994; Yonelinas et al., 1998). Specifically, Yonelinas (1994) assumed that OLD responses in the old–new recognition-memory task are a binary mixture. Some OLD responses occur because the participant recollected that the item was old and other OLD responses occurred because recollection failed but the item appeared highly familiar. Yonelinas (1994) further assumed that recollection operates as in classical threshold theory — that is, on target trials, there is some probability p that recollection is successful and on these trials the participant always responds OLD. If recollection fails, which occurs with probability $1 - p$, it completely fails in the sense that there is no partial recollection value that can be used to select a response. Instead, the choice between responding OLD versus NEW depends completely on familiarity. In the model proposed by Yonelinas (1994), judgments based on familiarity are modeled by the normal, equal-variance SDT model. The dual-process model predicts that

$$P(\text{OLD}|\text{target}) = p + (1 - p)\Phi(\mu_T - X_C), \quad (1)$$

where Φ is the standard-normal cumulative distribution function, μ_T is the mean familiarity strength of target items, and X_C is the criterion on familiarity for responding OLD. The model also assumes that recollection is impossible on lure trials, so the probability of responding OLD on lure trials – that is, the probability of a false alarm – is exactly the same as in the normal, equal variance SDT model. In other words,

$$P(\text{OLD}|\text{lure}) = 1 - \Phi(X_C) = \Phi(-X_C). \quad (2)$$

Note that this model has three free parameters: p, μ_T, and X_C — the same number as the normal, unequal variance model. It is well known that the ROC curves predicted by this model are skewed in the same direction as empirical ROC curves from the old–new recognition task (that is, when $p > 0$).

The mixture SDT model of recognition memory is similar to the dual-process model, in the sense that both models assume target trials include a mixture of responses that have different statistical properties (DeCarlo, 2002, 2010; Koen et al., 2017). The mixture SDT model assumes the two types of OLD responses on target trials are due to different levels of attention during the encoding of the target item at the time of initial study (DeCarlo, 2002). The idea is that as the study items are initially presented, the participant's attention waxes and wanes. The target items presented when attention is high will elicit a large memory strength when later presented during the testing phase, whereas target items initially presented when attention was low will elicit a weak memory strength during test.

The mixture SDT model (DeCarlo, 2002) models the memory strengths elicited by lures exactly as the noise distribution is modeled in the normal, equal-variance SDT model. As a result,

$$P(\text{OLD}|\text{lure}) = \Phi(-X_C). \quad (3)$$

In contrast, the model assumes that the memory strengths elicited by targets are a binary mixture of large and small memory strengths, each normally distributed with variance 1, but with different means. More specifically, the model assumes that

$$P(\text{OLD}|\text{target}) = p\Phi(d_1 - X_C) + (1 - p)\Phi(d_2 - X_C), \quad (4)$$

where p is the probability that a target item received a high level of attention during study. So d_1 is the mean memory strength when attention was high and d_2 is the mean memory strength when attention was low. In many applications, it is common to assume $d_2 = 0$, or in other words, that the mean memory strength during low attention is the same as for lures. According to this version of the model, with probability $1-p$ the participant completely ignores the target item during study. One advantage of this simplifying assumption is that it reduces the number of free parameters in the model from 4 to 3 (i.e., p, d_1, and X_C), and as a result, this version of the mixture model has the same number of free parameters as the unequal-variance and dual-process SDT models. The mixture model predicts ROC curves that are similar to the ROC curves predicted by the dual-process SDT model. This is not a surprise because both models make identical predictions for $P(\text{OLD}|\text{lure})$ and similar predictions for $P(\text{OLD}|\text{target})$ (i.e., compare Eqs. (1) and (4)).

3. The fixed-point property of binary mixtures

The normal, unequal-variance model, the dual-process model, and the mixture model all make similar, but not identical predictions about ROC curves in old–new recognition-memory experiments. For example, all of these models predict, or can predict, skewed ROC curves that generally have a slope in Z-ROC space that is less than 1.0 — properties that are present in virtually all empirical ROCs estimated from the old–new recognition task. The most striking difference is that the unequal-variance model predicts that all Z-ROC curves must be linear, whereas the mixture models predict that the Z-ROC curves become progressively more curved (i.e., more nonlinear) as the mixture probability p increases. Despite these similar predictions, the models make fundamentally different assumptions about the nature of processing on target trials. The normal, unequal-variance model predicts this is a unitary process in the sense that responding on every trial is mediated by the same psychological processes. In contrast, both the dual-process and mixture models assume that responding on target trials is a mixture of two qualitatively different types of trials. It turns out that this prediction that responding to targets is a mixture of two types of trials has a distinct empirical signature that, to my knowledge, has never been investigated. That signature is described in this section.

Mixture models have been proposed in a variety of different domains. For example, the fast-guess model of the speed-accuracy tradeoff accounts for fast error responses by proposing that observable response times (RTs) are a probability mixture of two types of responding (Yellott, 1971). With some probability p, the participant ignores the stimulus and just randomly guesses a response. The lack of perceptual or cognitive processing on these trials causes accuracy to be low and RT to be fast. With probability $1 - p$, the participant fully processes the stimulus, causing responding to be accurate and slow. Although better models of the speed-accuracy tradeoff were subsequently developed (e.g., see van Maanen, 2016), the fast-guess model is highlighted here because it remains one of the most conceptually simple and widely known mixture models.

All binary-mixture models have a unique empirical signature that was discovered by Falmagne (1968), which he called the *fixed-point property*. Specifically, consider a set of experimental conditions in which the mixture probability varies but the component distributions do not. For example, in the case of the fast-guess model, the conditions might vary speed stress. The fast-guess model predicts that the more that speed is emphasized, the more likely it is that the participant will ignore the stimulus and make a quick guess. So the model predicts that the primary effect of increasing speed stress will be to increase the guessing probability p. Falmagne (1968) showed that in any experiment with different conditions that affect the mixture probability p, but not the component distributions, all binary-mixture models predict that the probability density functions (pdfs) predicted for each experimental condition must all intersect at exactly the same point. This property is described more formally in the following result.

The Fixed-Point Property of Binary Mixtures. *Consider a model that predicts that some relevant pdf in experimental condition i, denoted*

$f_i(x)$, is a binary mixture of two component pdfs $g_1(x)$ and $g_2(x)$, with mixture probability p_i. More specifically, suppose that

$$f_i(x) = p_i\, g_1(x) + (1 - p_i)\, g_2(x). \tag{5}$$

Consider a set of conditions in which the mixture probability p_i varies, but $g_1(x)$ and $g_2(x)$ do not. Then if the pdfs predicted for each condition intersect, they must all intersect at the same fixed point.

Proof. The proof, which is straightforward, is due to Falmagne (1968). For convenience, it is reproduced here. Suppose there exists some value x^* where $g_1(x)$ and $g_2(x)$ intersect — that is, for which $g_1(x^*) = g_2(x^*)$. Now let p_i and p_j be any two mixture probabilities. Then note that

$$\begin{aligned}(p_i - p_j)\, g_1(x^*) &= (p_i - p_j)\, g_2(x^*)\\ &= [(1 - p_j) - (1 - p_i)]\, g_2(x^*),\end{aligned} \tag{6}$$

and therefore

$$p_i\, g_1(x^*) - p_j\, g_1(x^*) = (1 - p_j)\, g_2(x^*) - (1 - p_i)\, g_2(x^*). \tag{7}$$

Rearranging both sides produces

$$p_i\, g_1(x^*) + (1 - p_i)\, g_2(x^*) = p_j\, g_1(x^*) + (1 - p_j)\, g_2(x^*), \tag{8}$$

and therefore $f_i(x^*) = f_j(x^*)$ for any values of p_i and p_j. □

The fixed-point property holds for any random variable that is a binary mixture. As we saw, the mixture SDT model predicts that on target trials, the decision variable that drives old–new recognition judgments is a binary mixture of two normal distributions (i.e., see Eq. (4)). As a result, it predicts that across a set of conditions in which the mixture probabilities vary but the component pdfs do not, the set of all target distributions predicted by the mixture model must all intersect at the same point. The dual-process SDT model is also a binary mixture model but its predicted familiarity distributions are not constrained by the fixed-point property because there is no $g_1(x)$ pdf that intersects with $g_2(x)$. The big-picture question here though is why any of this should matter since memory-strength distributions are not observable. The fixed-point property is useful only if the mixture variable is some observable dependent measure. If it was observable, then a strong test of the model would be to estimate the mixture distributions and check whether they all intersect at the same point. Although memory strength is not an observable variable, there are several dependent measures that should depend directly on memory strength and therefore are candidates for tests of the fixed-point property. Two dependent variables come immediately to mind — RTs and ROC curves. The next two sections consider each of these possibilities in turn.

4. Response time tests of the fixed-point property

By themselves, none of the recognition-memory models considered above make any assumptions or predictions about the RTs that might be expected in the old–new recognition-memory task. Even so, there are generalizations of SDT that do make RT predictions. Among the oldest and simplest methods for deriving RT predictions from SDT is to add a straightforward assumption to SDT, called the RT-distance hypothesis, that simply assumes that RT decreases with the distance between the percept and the response criterion. The idea is that if decisions are made by comparing a percept or memory strength to a criterion, then the greater the distance between the two, the easier, and hence the faster the decision. Theoretical predictions of the generalized SDT model that includes this assumption were worked out by Murdock (1985). Empirical support for the RT-distance hypothesis was first reported by Emmerich et al. (1972) and Gescheider et al. (1969) and later, within the multidimensional context of general recognition theory, by Ashby et al. (1994). Furthermore, Murdock (1985) showed that this generalized SDT model gives good accounts of the RTs that are observed in recognition-memory experiments.

In addition, it is important to note that virtually all current process interpretations of SDT that make RT predictions are in general agreement with the RT-distance hypothesis, including for example, the drift–diffusion model (Ratcliff, 1978). The drift–diffusion model is a special case of a more general class of sequential-sampling models, which assume that the observer repeatedly samples the stimulus on each trial and then converts the sampled percepts into evidence favoring one of the two responses. Evidence is typically not defined explicitly in these models, but in a task where the two response alternatives are A and B, a natural definition of the strength of evidence associated with the percept $\underline{x}$ is

$$e(\underline{x}) = \left|\log\ L(\underline{x})\right| = \left|\log \frac{f_A(\underline{x})}{f_B(\underline{x})}\right| = \left|\log\ f_A(\underline{x}) - \log\ f_B(\underline{x})\right|. \tag{9}$$

According to this definition, there is no evidence that favors one response over the other when the likelihood ratio equals 1. Both responses are equally likely to be correct. As the likelihood ratio moves away from 1 – in either direction – the evidence about which response is correct increases. This definition is especially relevant to recognition memory because of the important role that likelihood ratio plays in recognition memory models (e.g., Glanzer et al., 2009). According to the reasonable Eq. (9) definition of evidence, note that if the likelihood ratio is monotonic with increases in sensory magnitude (or memory strength), then evidence increases with the distance from the percept to the response criterion X_C. So the RT predictions of any such sequential sampling model are in general agreement with the RT-distance hypothesis. As a result, adding the RT-distance hypothesis to any of the recognition-memory models considered above represents a theoretically minimal approach to extending these models to the RT domain.

The following result shows that adding the RT-distance hypothesis to any model that predicts memory strengths on target trials are a binary mixture causes that model to also predict that the RT pdfs on target trials must satisfy the fixed-point property.

Theorem 1. *Consider any recognition-memory model that assumes OLD responses in the old–new recognition task are a binary mixture — that is, any model described by Eq. (5) (e.g., including the dual-process and mixture SDT models). Suppose we extend this class of models to the RT domain by assuming that*

$$RT = T + T_0, \tag{10}$$

where T is the decision time that is computed by adding the RT-distance hypothesis to the memory model and T_0 is any motor time that is statistically independent of T. Now consider an old–new recognition-memory experiment that includes a set of conditions in which the mixture probability p_i varies across conditions, but the component distributions do not. Then all RT pdfs predicted by these models on target trials when the participant responds correctly must satisfy the fixed-point property — that is, if the RT pdfs predicted for each condition intersect, they must all intersect at the same fixed point.

Proof. The RT-distance hypothesis assumes that decision time $T = h(D)$, where h is a differentiable, strictly-decreasing function and D is the distance from the current percept to the response criterion X_C. All binary mixture models assume that the distribution of memory strengths on target trials equals

$$f(x|\text{target}) = pg_1(x) + (1 - p)g_2(x), \tag{11}$$

for some component pdfs g_1 and g_2. This model predicts that a correct OLD response occurs on trials when a random sample from this mixture distribution is greater than the criterion X_C. Let D denote the distance from this sample to the criterion. Then these distances will have one distribution on trials when the random sample comes from component distribution g_1 and a different distribution on trials when the random sample comes from component distribution g_2. Murdock

(1985) showed that if the component distribution is normal, then the distribution of distances to the criterion on correct response trials has a truncated normal distribution. However, the present theorem requires no distributional assumptions about g_1 or g_2, or about the specific function h that converts each distance to a decision time — that is, the theorem holds for any distributions g_1 and g_2 and any differentiable, strictly-decreasing function h.

If we denote the two distance distributions by g_{1D} and g_{2D}, respectively, then the distribution of distances across all correct OLD response trials equals

$$f_D(x|\text{target}) = p\, g_{1D}(x) + (1-p)\, g_{2D}(x). \tag{12}$$

The RT-distance hypothesis assumes decision time $T = h(D)$, where h is a differentiable strictly decreasing function. By the change-of-variable theorem of probability theory, the decision time pdf on target trials, denoted $f_T(t)$, equals

$$\begin{aligned} f_T(t) &= f_D\left[h^{-1}(t)\right]\left|\frac{dh^{-1}(t)}{dt}\right| \\ &= \left\{p\, g_{1D}\left[h^{-1}(t)\right] + (1-p)\, g_{2D}\left[h^{-1}(t)\right]\right\}\left|\frac{dh^{-1}(t)}{dt}\right| \\ &= p\left\{g_{1D}\left[h^{-1}(t)\right]\left|\frac{dh^{-1}(t)}{dt}\right|\right\} + (1-p)\left\{g_{2D}\left[h^{-1}(t)\right]\left|\frac{dh^{-1}(t)}{dt}\right|\right\} \\ &= p\, g_{1T}(t) + (1-p)\, g_{2T}(t), \end{aligned} \tag{13}$$

where $g_{1T}(t)$ and $g_{2T}(t)$ are the decision time pdfs on trials when the percept is a random sample from $g_1(x)$ and $g_2(x)$, respectively. As a result, $f_T(t)$ is a binary mixture and therefore satisfies the fixed-point property.

Note that Eq. (13) also holds for the dual-process model, except in this case $g_{1T}(t)$ is the decision-time pdf on trials when recollection succeeds. This pdf does not depend on distance-to-criterion, and instead is the same on all target trials and in all conditions. The key point though is that the dual-process model also predicts that the decision times on correct target trials are a binary mixture. With probability p, decision time is a random sample from the recollection successful distribution [i.e., from $g_{1T}(t)$] and with probability $1-p$, decision time is a random sample from the $g_{2T}(t)$ distribution. Therefore, the dual-process model also predicts that decision times must satisfy the fixed-point property.

Suppose now that a random motor time T_0 is added to each decision time, so the observable RT equals $RT = T + T_0$. Suppose also that T and T_0 are statistically independent and let $f_0(t)$ denote the pdf of T_0. Then the RT pdf equals

$$\begin{aligned} f_{\text{RT}}(t) &= f_T(t) * f_0(t) \\ &= \left[p\, g_{1T}(t) + (1-p)\, g_{2T}(t)\right] * f_0(t) \\ &= p\left[g_{1T}(t) * f_0(t)\right] + (1-p)\left[g_{2T}(t) * f_0(t)\right], \end{aligned} \tag{14}$$

where $*$ denotes convolution. The last equality holds because convolution is a linear operation and therefore satisfies distributivity. Since the convolution of two pdfs is a pdf, it therefore follows that the fixed-point property still holds if an independent base time or motor time is added to the decision time on each trial. □

A test of this strong prediction of all binary-mixture models requires an old–new recognition-memory experiment that includes at least three conditions that the model predicts should all be identical, except for the numerical value of the mixture probability p_i. The exact design of this experiment might depend on which model is being tested. The dual-process model assumes that p is the probability that recollection is successful, so a candidate experiment would manipulate some independent variable that selectively influences the probability of recollection. In contrast, the mixture model assumes p is the probability that the item was attended to during study, so in this case, a candidate experiment would manipulate an independent variable that selectively influences attention. There are likely a variety of ways to design such an experiment, but in the case of the mixture model, one possibility might be as follows. The goal is to manipulate an independent variable that causes the amount of attention available to the participant for encoding the target list during initial study to vary across conditions. An obvious possibility is to require the participant to perform a simultaneous dual task during encoding that varies across conditions in memory load. For example, the experiment might include four conditions in which the participant is required to study the target items while holding in working memory a list of 0, 2, 4, or 6 digits, respectively. Previous research suggests that this design should cause reduced encoding of the target items with increased memory load of the dual task (e.g., Jolicoeur, 1999).

Theorem 1 is illustrated in Fig. 1, which shows RT pdfs predicted by the dual-process SDT model [panel (a)], the mixture SDT model [panel (b)], and the normal, unequal-variance SDT model [panels (c) and (d)] in a hypothetical experiment of this type. In all cases, predicted RT pdfs are shown for correct OLD responses in four different hypothetical experimental conditions. In all cases where the decision was based on an SDT model in which the memory strength was compared to a response criterion X_C, RT was computed from an RT-distance model in which decision time was a power function of the distance between the memory strength and X_C. Specifically, in all cases

$$RT = T_0 + 450\, D^{-0.35}, \tag{15}$$

where T_0 was a normally distributed motor time with mean 100 ms and standard deviation 10 ms.

Consider first the dual-process SDT model with predictions shown in panel (a). These predictions were generated by setting $\mu_\text{T} = 1.5$ (i.e., see Eq. (1)). On trials when the OLD response was determined by recollection, I assumed that RT had an exGaussian distribution (Hohle, 1965; Ratcliff, 1978), which is among the most popular current models of RT pdfs. The exGaussian distribution is the distribution that results when independent random samples from a normal distribution and an exponential distribution are added together. In this case, I assumed the normal distribution mean and standard deviation were 350 ms and 10 ms, respectively, and that the mean of the exponential distribution was 125 ms. As a result, the mean on all recollection trials was 475 ms (350 + 125), regardless of familiarity. Fig. 1a shows predictions of the model for four different values of p_i (i.e., the probability of successful recollection). Note that, as the theorem requires, all four pdfs intersect at the same fixed point.

Fig. 1b shows similar predictions for the mixture SDT model. The parameters were all set to the same values as in panel (a), except the means of the two component target distributions were set to 0.5 and 2.0 (i.e., d_1 and d_2 in Eq. (4)). Note again that, as required by the theorem, all four distributions pass through the same fixed point.

Panels (c) and (d) of Fig. 1 show that this fixed-point property is not a feature that should be expected to hold in models that are not constructed from a binary mixture. Figs. 1c and d show predictions of the normal, unequal-variance model for this same hypothetical experiment. In panel (c), the mean of the target distribution is assumed to change across conditions, whereas panel (d) was produced under the assumption that the variance of the target distribution changed. Note that these pdfs all intersect, but each pair intersect at a different time point, and therefore do not satisfy the fixed-point property.

In Fig. 1 application, the normal, unequal-variance model has 3 free parameters for each pdf (i.e., the mean and variance of the target distribution, and the response criterion). Therefore, for the 4 pdfs shown in Figs. 1c and 1d, a total of 12 free parameters could be manipulated. In other words, the parameter space for the normal, unequal-variance model in this application is 12 dimensional. Each point in this 12-dimensional space generates a different set of 4 pdfs. There is little doubt that at least some points in this space would cause the model to generate pdfs that all intersect at the same point. Consider any one of such points. Now suppose we move in the 12-dimensional parameter space in any direction from this point. Any such movement

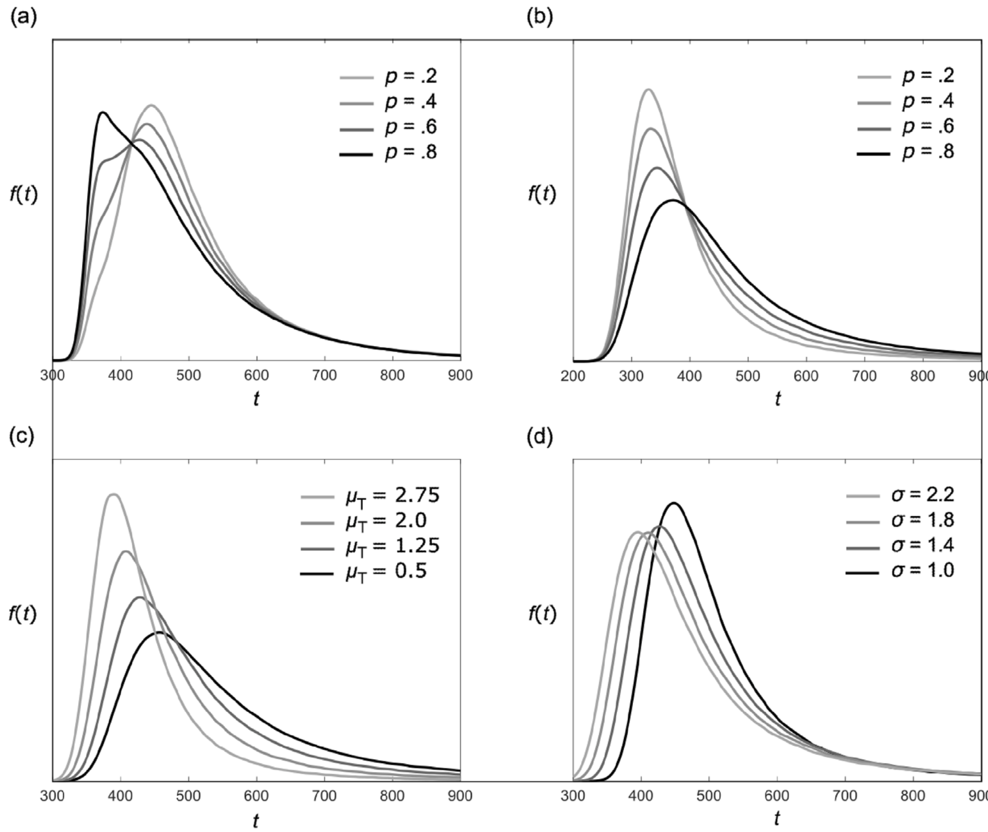

Fig. 1. RT pdfs predicted by recognition-memory models on target trials of an old–new recognition-memory task on which the participant responds correctly. (a) Predictions of the dual-process model when the only change across conditions is in the value of p. (b) Predictions of the mixture model when the only change across conditions is in the value of p. (c) Predictions of the normal, unequal-variance model when the only change across conditions is in the mean of the target distribution. (c) Predictions of the normal, unequal-variance model when the only change across conditions is in the variance of the target distribution.

will cause the model to predict a change in one or more of the pdfs. Because the model predicts that the fixed-point property occurs because of coincidence, rather than because of any structural property of the model, moving in any direction from our hypothetical point should cause the fixed-point property to fail. As a result, we expect that the normal, unequal-variance model can account for fixed-point pdfs only at some set of discrete points in its 12-dimensional parameter space. Such a set has measure 0, regardless of how many points it contains, and therefore if the prior distributions on the 12 parameters are continuous, then the probability that the model predicts fixed-point pdfs a priori equals 0. In contrast, the dual-process and mixture models predict that the 4 pdfs will all cross at the same point for every point in their parameter spaces, and therefore the models predict that the a priori probability of finding a fixed point equals 1 (i.e., assuming the parameter space is restricted to regions where the pdfs intersect at least once). A well-established model selection criterion is to favor a model that predicts an observed result a priori over a model that makes no such prediction, but in which some set of parameter values can be found that allow the model to account for the result post hoc. Even so, of course, no single psychological experiment is definitive, so any finding of a fixed point should be replicated and generalized to other experimental conditions.

It should be stressed that this test of mixture models is parameter free. By this I mean that the test is valid for any version of the models — no matter how the models are parameterized and for a given parameterization, no matter how many parameters are allowed to be free. For example, consider the mixture model. As mentioned earlier, to reduce the number of free parameters in this model, it is common to assume that one component target distribution is normal with mean d_1 and variance 1, whereas the other is normal with mean 0 and variance 1. This version of the model has three free parameters (i.e., p, X_C, and d_1), which is the same number as the normal, unequal variance model. As a result, the goodness-of-fit of the two models to empirical ROC data can be compared directly. But Theorem 1 holds for any version of the mixture model. So a generalized version in which one component target distribution is normal with mean d_1 and variance σ_1^2 and the other is normal with mean d_2 and variance σ_2^2 still predicts that all RT pdfs from the experiment described in Theorem 1 must intersect at the same time point, even though the three extra parameters in this version of the model (i.e., d_2, σ_1^2, and σ_2^2) would allow it to provide good fits to a much more diverse set of empirical ROC data.

To my knowledge, this differential prediction of the dual-process and mixture models versus the normal, unequal-variance model has never been empirically investigated. It seems worthy of empirical test however, because it is such a strong prediction. A finding that all empirical RT pdfs cross at the same time point seems like strong evidence that OLD responses are a binary mixture of two different trial types because it seems unlikely that a model postulating a unitary process would coincidentally predict a set of pdfs that satisfy this condition. Furthermore, note that the dual-process and mixture SDT models also make the strong prediction that this fixed-point prediction should fail on lure trials, since both models predict that memory strengths elicited by lures are not binary mixtures (i.e., see Eqs. (2) and (3)). Fortunately, a statistical test of the fixed-point property has been developed, as well as an R package that implements this test (van Maanen et al., 2014). On the other hand, the fixed-point property could fail for a variety of different reasons, even if OLD responses are a binary mixture (van Maanen et al., 2016). The most likely scenario is probably that the experimental manipulations caused one or both of the component distributions to change across conditions. For these reasons, a failure of the fixed-point property should be interpreted with caution.

5. ROC curve tests of the fixed-point property

A qualitatively different possibility is that a binary-mixture target distribution will leave an empirical signature in the ROC curves that it produces. An advantage of an ROC-curve test of mixture models is that no extra assumptions about processing time are needed. For example, there is no need to assume the RT-distance hypothesis. As this subsection shows, a large class of mixture models do make a striking ROC curve prediction. This prediction is described in the next result.

Theorem 2. *Consider an old–new recognition-memory experiment with a variety of different conditions. Now consider a model that assumes a single lure distribution and that the target distribution is a binary mixture, that is, in which*

$$f_L(x) = g_0(x), \tag{16}$$

and

$$f_T(x) = p_i\, g_1(x) + (1 - p_i)\, g_2(x), \tag{17}$$

where p_i is the mixture probability in condition i. Suppose this model satisfies the following conditions:

(1) The mixture probability p_i varies across conditions, but the lure distribution and the two component target distributions do not.
(2) The cumulative distribution functions G_1 and G_2 of the two component target distributions cross — that is, there exists some value x^* for which $G_1(x^*) = G_2(x^*)$.

Then all ROC curves predicted by this model must intersect at the same point in ROC space.

Proof. Let $P_i(\text{OLD}|\text{target})$ and $P_i(\text{OLD}|\text{lure})$ denote the probabilities of responding OLD on target and lure trials in condition i, respectively. First, note that under the conditions specified in the theorem, when the criterion to respond OLD equals $X_C = x^*$ then

$$P_i(\text{OLD}|\text{lure}) = 1 - F_L(x^*) = K^*, \tag{18}$$

for some constant K^*, and where F_L is the cumulative distribution function of the lure distribution. In other words, when $X_C = x^*$, the probability of a false alarm is the same in all conditions. Second, note that Eq. (17) implies that the target cumulative distribution function is also a binary mixture. Specifically, note that integrating both sides of Eq. (17) leads to

$$F_T(x) = p_i\, G_1(x) + (1 - p_i)\, G_2(x). \tag{19}$$

Third, it is well known and straightforward to show that the fixed-point property also holds for cumulative distribution functions.[1] Specifically, if the two component cumulative distribution functions cross at some point x^*, then all cumulative target distribution functions predicted by this model also cross at x^*. As a result,

$$F_T(x^*|\text{condition}\, i) = F_T(x^*|\text{condition}\, j) = 1 - C^*, \tag{20}$$

for any conditions i and j, and for some constant C^*. Therefore, note that

$$P_i(\text{OLD}|\text{target}) = 1 - F_T(x^*|\text{condition}\, i) = C^*, \tag{21}$$

for all i. In other words, when $X_C = x^*$, $P(\text{OLD}|\text{target})$ is also the same in all conditions. As a result, the ROC curve must pass through the point (K^*, C^*) in all conditions. □

Theorem 2 says that if the component cumulative distribution functions of a mixture model cross, then the ROC curves predicted by that model must also all cross at the same point in ROC space.

Theorem 2 is illustrated in Fig. 2. Panels (a) and (b) show ROC curves for two different mixture models, and panels (c) and (d) show the same two curves, except plotted in Z space. In all cases, the lure distribution is normal with mean 0 and variance 1, the means of the two component target distributions are $d_1 = 1.5$ and $d_2 = 1$, and the standard deviation of component distribution 2 is $\sigma_2 = 1.1$. In panels (a) and (c), the first component distribution has standard deviation $\sigma_1 = 1.4$, whereas $\sigma_1 = 1.7$ in panels (b) and (d). Within each panel, the different curves were generated from different values of the mixture probability p (i.e., p = .1, .4, .65, and .9). Because $\sigma_1 \neq \sigma_2$, this model predicts that the cumulative distribution functions of the target component distributions cross, and therefore the conditions of Theorem 2 are met, and as a result, the ROCs must cross.

There are several points of note here. First, as comparison, note that the equal-variance mixture model (see DeCarlo, 2002) and the dual-process model (see Yonelinas, 1994) both predict Z-ROC curves that are nonlinear and ordered by the value of the mixture probability p (and therefore do not intersect). Second, the unequal variances in the two component distributions of the mixture model used to generate Fig. 2 are consistent with the expectation that in nature, variance increases with mean. In Fig. 2, the lure, target component 2, and target component 1 distributions have means equal to 0, 1, and 1.5, respectively and standard deviations equal to 1, 1.1, and either 1.4 or 1.7, respectively. So in all cases, the variance increases with the mean. As a result, the model illustrated in Fig. 2 can be seen as a melding of the (equal-variance) mixture model (DeCarlo, 2002) and the normal, unequal-variance SDT model in which familiarity and recollection have additive effects on memory strength (Rotello et al., 2004; Wixted & Stretch, 2004).

Third, it is important to note that although Theorem 2 guarantees that the ROC curves will all intersect at the same point if the component distributions have different variances, this prediction may have limited empirical utility if the variance difference is small. This is because the intersection will occur so many standard deviations out in Z space that it will essentially be impossible to test empirically. As a result, a failure to find that a set of ROC curves all intersect at the same point says little about whether the target distribution is a binary mixture. On the other hand though, if the ROC curves do all intersect at the same point, then a mixture model might be strongly suspected because this is not a feature of ROC curves that is predicted by non-mixture models.

Theorem 2 does not depend on any processing-time assumptions, but it does not hold for all mixture models. Specifically, it does not hold for any model in which the likelihood ratio $g_2(x)/g_1(x)$ is monotonic in x. For all such models, Condition 2 of Theorem 2 is violated. This is because a monotonic likelihood ratio guarantees that the cumulative distribution functions are ordered (Townsend & Ashby, 1983, p. 281). As a result, Condition 2 can only be met if the g_2/g_1 likelihood ratio is nonmonotonic. Note that this implies that the mixture model proposed by DeCarlo (2002) (described by Eqs. (3) and (4)) in which g_1 and g_2 are both normal with equal variance does not predict intersecting ROC curves (because the likelihood ratio of two normal distributions with equal variance is monotonic). Even so, more general versions of the mixture model have been proposed in which the two component target distributions have different variances (Koen et al., 2017). Theorem 2 applies to these models.

It is also important to note that Theorem 2 makes no assumption about the lure distribution. Specifically, it does not require nonmonotonicity of the target-to-lure likelihood ratio $f_T(x)/f_L(x)$. Therefore, it does not assume that extremely low familiarities are more likely on target trials than on lure trials, as predicted for example, by the unequal-variance SDT model. It is also worth noting though, that although the unequal-variance SDT model does make this prediction, the

[1] To see this, simply recreate the proof of the fixed-point property, except substitute the cumulative distribution functions G_1 and G_2 for the pdfs g_1 and g_2.

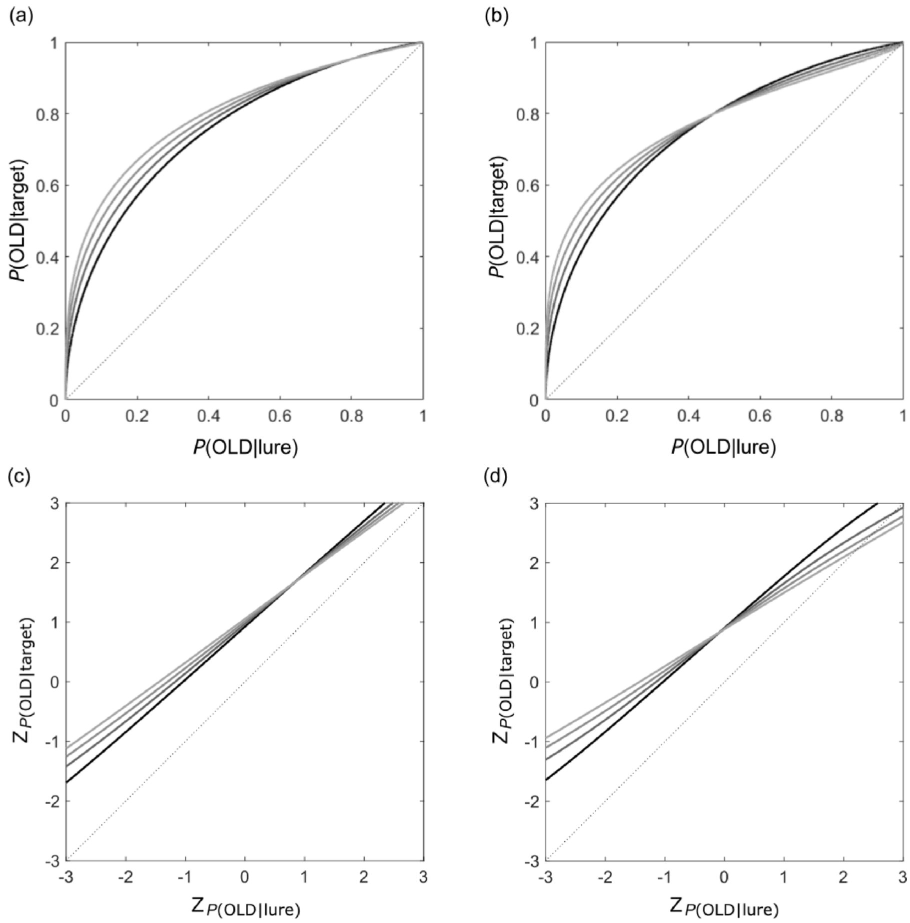

Fig. 2. ROC curves predicted by the unequal-variance mixture SDT model of recognition memory. In all cases, $d_1 = 1.5$, $d_2 = 1$, and $\sigma_2 = 1.1$. The curves differ only in the value of the mixture probability (i.e., p = .1, .4, .65, and .9). (a) ROC curves predicted by the model when $\sigma_2 = 1.4$. (b) ROC curves predicted by the model when $\sigma_2 = 1.7$. (c) Same curves as in panel (a), except plotted in Z space. (d) Same curves as in panel (b), except plotted in Z space.

model also predicts that in typical experiments these low familiarities are essentially impossible on target trials. As mentioned earlier, a representative value for the slope of the Z-ROC in recognition memory experiments is about 0.8, which is consistent with an unequal-variance SDT model in which the target standard deviation is 1.25, given a lure standard deviation of 1.0. Assuming a target mean of 1.5, then given these standard deviations, a low familiarity is more likely from the target distribution only when its strength is less than −6.3. According to this model, familiarities of target items are this low only with a probability of less than 0.000001, and therefore the model predicts that this outcome is essentially impossible.

6. Conclusions

The normal, unequal-variance model, the dual-process model, and the mixture model all make similar predictions about ROC curves in old–new recognition-memory experiments. This has made them difficult to discriminate empirically. This note showed that despite their similar ROC predictions, the dual-process and mixture models make some striking predictions that the normal, unequal-variance model does not make. Specifically, in any experiment that includes conditions in which the mixture probability varies but the component distributions do not, the dual-process and mixture models predict that all RT pdfs (and cdfs) must intersect at the same time point (if they intersect at all). And similarly, both models predict that if the ROC curves from this experiment intersect, they must also all intersect at the same point. Note that this RT prediction is more general than the ROC curve prediction because the prediction of a fixed point in the ROC curves requires the extra assumption that the cumulative distribution functions of the two component target distributions cross, whereas all binary-mixture models predict a fixed point in the RT pdfs (assuming the RT-distance hypothesis is valid).

It is important to note that empirical tests of these predictions that yield positive results are much more informative than tests that yield negative results (although care must be taken to guard against false positives; Couto et al., 2024). This is because all binary-mixture models predict that there are a variety of different ways that the fixed-point prediction could fail. The most obvious example is that the independent variable that was manipulated to create the various experimental conditions affects the component distributions in some way, causing one or both of them to vary across conditions. On the other hand, a positive result seems like strong evidence in favor of some binary-mixture model. Although it is logically possible that some single-process model could coincidentally produce RT pdfs or ROC curves that all cross at the same point, such an outcome would be just that — a coincidence, and a rather large one at that.

CRediT authorship contribution statement

F. Gregory Ashby: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization.

References

Ashby, F. G., Boynton, G., & Lee, W. W. (1994). Categorization response time with multidimensional stimuli. *Perception & Psychophysics*, *55*(1), 11–27.

Couto, J., Lebreton, M., & van Maanen, L. (2024). Specificity and sensitivity of the fixed-point test for binary mixture distributions. *Behavior Research Methods*, *56*(4), 2977–2991.

DeCarlo, L. T. (2002). Signal detection theory with finite mixture distributions: Theoretical developments with applications to recognition memory. *Psychological Review*, *109*(4), 710–721.

DeCarlo, L. T. (2010). On the statistical and theoretical basis of signal detection theory and extensions: Unequal variance, random coefficient, and mixture models. *Journal of Mathematical Psychology*, *54*(3), 304–313.

Emmerich, D. S., Gray, J. L., Watson, C. S., & Tanis, D. C. (1972). Response latency, confidence, and ROCs in auditory signal detection. *Perception & Psychophysics*, *11*(1), 65–72.

Falmagne, J.-C. (1968). Note on a simple fixed-point property of binary mixtures. *British Journal of Mathematical and Statistical Psychology*, *21*(1), 131–132.

Gescheider, G. A., Wright, J. H., Weber, B. J., Kirchner, B. M., & Milligan, E. A. (1969). Reaction time as a function of the intensity and probability of occurrence of vibrotactile signals. *Perception & Psychophysics*, *5*, 18–20.

Glanzer, M., Hilford, A., & Maloney, L. T. (2009). Likelihood ratio decisions in memory: Three implied regularities. *Psychonomic Bulletin & Review*, *16*(3), 431–455.

Glanzer, M., Kim, K., Hilford, A., & Adams, J. K. (1999). Slope of the receiver-operating characteristic in recognition memory. *Journal of Experimental Psychology. Learning, Memory, and Cognition*, *25*(2), 500–513.

Hohle, R. H. (1965). Inferred components of reaction times as functions of foreperiod duration. *Journal of Experimental Psychology*, *69*(4), 382–386.

Jolicoeur, P. (1999). Dual-task interference and visual encoding. *Journal of Experimental Psychology: Human Perception and Performance*, *25*(3), 596–616.

Koen, J. D., Barrett, F. S., Harlow, I. M., & Yonelinas, A. P. (2017). The ROC Toolbox: A toolbox for analyzing receiver-operating characteristics derived from confidence ratings. *Behavior Research Methods*, *49*(4), 1399–1406.

Murdock, B. B. (1985). An analysis of the strength-latency relationship. *Memory & Cognition*, *13*(6), 511–521.

Ratcliff, R. (1978). A theory of memory retrieval. *Psychological Review*, *85*(2), 59–108.

Ratcliff, R., Sheu, C.-F., & Gronlund, S. D. (1992). Testing global memory models using ROC curves. *Psychological Review*, *99*(3), 518–535.

Rotello, C. M., Macmillan, N. A., & Reeder, J. A. (2004). Sum-difference theory of remembering and knowing: A two-dimensional signal-detection model. *Psychological Review*, *111*(3), 588–616.

Townsend, J. T., & Ashby, F. G. (1983). *Stochastic modeling of elementary psychological processes*. Cambridge University Press.

van Maanen, L. (2016). Is there evidence for a mixture of processes in speed-accuracy trade-off behavior? *Topics in Cognitive Science*, *8*(1), 279–290.

van Maanen, L., Couto, J., & Lebreton, M. (2016). Three boundary conditions for computing the fixed-point property in binary mixture data. *PLoS ONE*, *11*(11), Article e0167377.

van Maanen, L., de Jong, R., & van Rijn, H. (2014). How to assess the existence of competing strategies in cognitive tasks: A primer on the fixed-point property. *PLoS ONE*, *9*(8), Article e106113.

Wixted, J. T. (2007). Dual-process theory and signal-detection theory of recognition memory. *Psychological Review*, *114*(1), 152–176.

Wixted, J. T., & Stretch, V. (2004). In defense of the signal detection interpretation of remember/know judgments. *Psychonomic Bulletin & Review*, *11*, 616–641.

Yellott, J. I. (1971). Correction for fast guessing and the speed-accuracy tradeoff in choice reaction time. *Journal of Mathematical Psychology*, *8*(2), 159–199.

Yonelinas, A. P. (1994). Receiver-operating characteristics in recognition memory: Evidence for a dual-process model. *Journal of Experimental Psychology. Learning, Memory, and Cognition*, *20*(6), 1341–1354.

Yonelinas, A. P., Kroll, N. E., Dobbins, I., Lazzara, M., & Knight, R. T. (1998). Recollection and familiarity deficits in amnesia: Convergence of remember-know, process dissociation, and receiver operating characteristic data. *Neuropsychology*, *12*(3), 323–339.