Counting and Timing Models in Psychophysics and the Conjoint Weber’s Law

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J. C. Falmagne, G. Iverson, and S. Marcovici (1979). Psychological Review, 86, 25-43) proposed a generalization of Weber’s law, which they called the conjoint Weber’s law. Empirically, the law sometimes holds. When it fails, the data satisfy a relation that Falmagne et al. identify as the conjoint Weber’s inequality. This paper investigates the ability of counting and timing models of psychophysics to predict Weber’s law and the conjoint Weber’s law. It is shown that although the timing model naturally predicts both laws to hold, all reasonable counting models predict them to fail. Instead, counting models naturally predict Weber’s inequality and the conjoint Weber’s inequality.

In a study of the binaural loudness summation hypothesis, Falmagne et al. (1979) proposed a generalization of Weber’s law, which they called the conjoint Weber’s law (see also Falmagne, 1985; Falmagne & Iverson, 1979), and they also reported results from an experiment supporting the law. Falmagne and his colleagues emphasize a measurement theoretic approach. Given that empirical support for the conjoint Weber’s law exists, it is of interest to ask whether popular processing models of psychophysics predict this law to be true.

Consider an experiment in which stimuli are constructed from two separate components differing, say, only in intensity. As a concrete example, assume the stimuli are auditory. In this case, stimulus \( ax \) is a pair of pure tones; one of intensity \( a \) input to the left ear and one of intensity \( x \) input to the right ear. The subject’s task is to compare stimulus \( ax \) with stimulus \( by \) and to indicate which is louder. Let \( P_{ax, by} \) be the probability that the subject chooses stimulus \( ax \) as loudest. Now suppose we multiply the intensity of each component of both stimuli by a constant, \( \lambda \). Then the probability that the subject chooses the stimulus created by multiplying the intensity of each component of \( ax \) by a constant is \( P_{(\lambda a)(\lambda x), (\lambda b)(\lambda y)} \). The conjoint Weber’s law states that

\[
P_{(\lambda a)(\lambda x), (\lambda b)(\lambda y)} = P_{ax, by}
\]

for every \( \lambda > 0 \). A single channel version of Weber’s law can be obtained as a special case by assuming that the intensities of the tones input to one ear, say the left, are zero. Thus this more traditional form of Weber’s law states that (e.g., Falmagne, 1985)

\[
P_{ax, ay} = P_{x, y}.
\]

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Although Falmagne et al. (1979) report data that support the conjoint Weber's law, Falmagne and Iverson (1979) indicate that in some of their unpublished data

\[
P_a(\lambda x) \geq P_{ax, bx}
\]

whenever \( a \geq b \) and \( x \geq y \) and for all \( \lambda \geq 1 \). They call this relation the conjoint Weber's inequality. Note that it is consistent with the fairly common finding of a decreasing Weber fraction (Falmagne, 1977; McGill & Goldberg, 1968).

Two popular models of intensity discrimination are the counting model (e.g., Audley & Pike, 1965; McGill, 1963, 1967; Pike, 1966, 1968, 1971, 1973; Townsend & Ashby, 1983; Vickers, 1979) and the timing model (Green & Luce, 1967, 1971, 1973; Luce & Green, 1970, 1972). Both of these assume that perceived intensity is determined by the sequence of neural impulses generated over parallel channels after presentation of the stimulus. In the timing model, perceived intensity is a function of the elapsed time from stimulus onset until a criterion number of impulses has occurred and in the counting model it is a function of the number of impulses that occur before a fixed criterion time has elapsed. Both models make many appealing predictions and in general they have been difficult to discriminate empirically (however, see Green & Luce, 1973).

Although Falmagne et al. (1979) did not explicitly consider whether either of these models predict the conjoint Weber's law, their results make it clear that, under very plausible assumptions about the form of the psychophysical function, the timing model does predict the conjoint Weber's law to hold. In this paper, I summarize their argument and then show that no reasonable counting model predicts the conjoint Weber's law to hold. Instead, counting models naturally predict the conjoint Weber's inequality. Further, each of these results specializes to the single channel Weber's law. Thus, Weber's law and the conjoint Weber's law are powerful tools for discriminating between counting and timing strategies.

One reason these two models are so difficult to discriminate is that they assume the same neural representation. Both models assume that judgments of sensory magnitude depend on a sequence of neural impulses arriving at some central decision center. The most popular versions assume that the train of neural spikes can be described by a Poisson process. Consider a trial on which stimulus \( ax \) is presented. Call the process generated on the left auditory channel \( L_a(t) \) and the process generated on the right channel, \( R_x(t) \), and let the corresponding rates be denoted by \( l(a) \) and \( r(x) \), respectively. In the types of experiments for which the conjoint Weber's law might be tested, it is plausible to assume that \( L_a(t) \) and \( R_x(t) \) are statistically independent.

Falmagne et al. (1979) associate the conjoint Weber's law with additivity between channels. With auditory stimuli this assumption is known as the binaural loudness summation hypothesis. It suggests that loudness information regarding the binaural tone \( ax \) is based upon the sum

\[
J_{ax}(t) = L_a(t) + R_x(t).
\]
Under these assumptions $J_{ax}(t)$ is a Poisson process with rate $l(a) + r(x)$. We now consider specific predictions of the timing and counting models.

**The Timing Model**

The timing model assumes that the subject makes judgments of sensory magnitude on the basis of the mean interspike arrival times. Following Falmagne et al. (1979), call this mean value $S_n(ax)$, where $n$ is the total number of spikes observed. Then

$$P_{ax, by} = P[S_n(by) - S_n(ax) > 0].$$

(1)

For large $n$ the mean interarrival times will be approximately normally distributed with mean

$$m(ax) = [l(a) + r(x)]^{-1}$$

and variance

$$\sigma^2(ax) = [l(a) + r(x)]^{-2}/n.$$  

Falmagne et al. (1979) show that these facts imply that for large $n$, Eq. (1) reduces to

$$P_{ax, by} = \Phi \left\{ \sqrt{n} f \left[ \frac{l(a) + r(x)}{l(b) + r(y)} \right] \right\},$$

(2)

where $\Phi$ is the standard normal cumulative distribution function and

$$f(s) = \frac{s - 1}{\sqrt{s^2 + 1}}, \quad \text{with} \quad s > 0.$$

Equation (2) is an example of what Falmagne et al. (1979) call a logarithmic difference PACOME (probabilistic additive conjoint measurement) model. As such, note that if $l$ and $r$ are strictly increasing and $y$ is set equal to $x$ then the timing model predicts that $P_{ax, bx}$ approaches $\frac{1}{2}$ as $x$ increases. If $a > b$, then $P_{ax, bx}$ decreases in $x$. This prediction is supported by the data of an experiment reported by Falmagne et al.

From Eq. (2) it is also clear that the conjoint Weber's law holds if and only if

$$\frac{l(\lambda a) + r(\lambda x)}{l(\lambda b) + r(\lambda y)} = \frac{l(a) + r(x)}{l(b) + r(y)}.$$

(3)

This is a functional equation. Falmagne et al. (1979) note that Eq. (3) is satisfied when $l$ and $r$ are power functions with the same exponent. The following result is a summary of their findings.
Theorem 1. If the subject employs a timing strategy and the conjoint Weber’s law holds and if \( l \) and \( r \) are continuous and \( l(0) = r(0) = 0 \), then
\[
l(x) = ax^b
\]
and
\[
r(x) = \gamma x^b,
\]
where \( x > 0 \), and \( a \), \( b \), and \( \gamma \) are constants with \( a \) and \( \gamma \) nonzero.

Proof. Set \( x = y = 0 \) and \( a = 1 \) in Eq. (3). Then after some simplification Eq. (3) can be written as
\[
l(\hat{x}, b) = \frac{l(\hat{x})}{l(1)} \frac{l(b)}{l(1)},
\]
or equivalently as
\[
\frac{l(\hat{x}, b)}{l(1)} = \frac{l(\hat{x})}{l(1)} \frac{l(b)}{l(1)}.
\]
Defining \( f(x) = l(x)/l(1) \) simplifies this expression to
\[
f(\hat{x}, b) = f(\hat{x}) f(b).
\]
This is a variant of the Cauchy equation. If \( f \) (and hence \( l \)) is continuous then its solution is (e.g., Aczel, 1966)
\[
f(x) = x^\beta
\]
for some constant \( \beta \). However, \( f(x) = l(x)/l(1) \) and so
\[
l(x) = ax^b,
\]
where \( a \) equals the nonzero constant \( l(1) \). A similar analysis for \( r(x) \) yields
\[
r(x) = \gamma x^b.
\]
An examination of Eq. (2) indicates that equality is satisfied only when \( \delta = \beta \).

The condition that \( l(0) = r(0) = 0 \) is necessary since Eq. (3) is satisfied, for example, when \( l(t) = \log t \) and \( r(t) = -\log t \). Although the issue of whether the two auditory channels are characterized by the same two exponents is unresolved (Falmagne et al., 1979; Hellman & Zwislocki, 1963; Levet et al. 1972; Reynolds & Stevens, 1960), the conditions under which the timing model predicts the conjoint Weber’s law to hold seem reasonable. Thus, empirical support of the conjoint Weber’s law provides indirect support of the timing model.
The Counting Model

The counting model assumes that the subject makes judgments of sensory magnitude on the basis of the mean number of spikes recorded during each unit observation interval. Suppose the duration of a unit interval is $\Delta t$ time units. Then let $S_{\alpha, \Delta t}(ax)$ be the mean number of spikes recorded during $n$ intervals when stimulus $ax$ is presented. Alternatively, $S_{\alpha, \Delta t}(ax)$ might be the mean number of spikes recorded on $n$ fibers during a single observation interval of length $\Delta t$. By assumption, the number of spikes in one interval of length $\Delta t$ is Poisson distributed with mean $[l(a) + r(x)] \Delta t$. Therefore, for large $n$, $S_{\alpha, \Delta t}(ax)$ is approximately normally distributed with mean

$$m(ax) = [l(a) + r(x)] \Delta t$$

and variance

$$\sigma^2(ax) = \{[l(a) + r(x)] \Delta t\}/n.$$ 

Now

$$P_{ax, by} = P[S_{\alpha, \Delta t}(ax) - S_{\alpha, \Delta t}(by) > 0]$$

$$= P\left[Z < \frac{m(ax) - m(by)}{\sqrt{\sigma^2(ax) + \sigma^2(by)}} \right].$$

After some simplification, this expression can be reduced to

$$P_{ax, by} = \Phi \left[\sqrt{n \Delta t} \frac{[l(a) + r(x) - l(b) - r(y)]}{\sqrt{l(a) + r(x) + l(b) + r(y)}} \right]. \quad (5)$$

First, note that, unlike the timing model, the counting model is not a logarithmic difference PACOME. However, it is what Falmagne et al. call a simple PACOME, since Eq. (5) can be written in the form

$$P_{ax, by} = F[l(a) + r(x), l(b) + r(y)],$$

where

$$F(r, s) = \Phi \left[\sqrt{n \Delta t} \frac{r - s}{\sqrt{r + s}} \right]$$

and thus $F$ is a continuous function strictly increasing in the first argument and strictly decreasing in the second.

When $y$ is set equal to $x$ and $a > b$, not all simple PACOMEs predict that $P_{ax, bx}$ is a decreasing function of $x$ (Falmagne et al., 1979). However, note that if $y = x$, then Eq. (5) simplifies to

$$P_{ax, bx} = \Phi \left[\sqrt{n \Delta t} \frac{l(a) - l(b)}{\sqrt{l(a) + l(b) + 2r(x)}} \right].$$
Therefore, if \( l \) and \( r \) are strictly increasing, the counting model predicts that \( P_{ax, bx} \) approaches \( \frac{1}{2} \) as \( x \) increases, and further, if \( a > b \), that \( P_{ax, bx} \) decreases in \( x \). These predictions agree with the timing model and since this is the empirical finding (Falmagne et al., 1979), both models make the correct prediction.

To determine whether the counting model is able to predict the conjoint Weber’s law, note that Eq. (5) indicates that the conjoint Weber’s law holds if and only if

\[
\frac{l(\lambda a) + r(\lambda x) - l(\lambda b) - r(\lambda y)}{\sqrt{l(\lambda a) + r(\lambda x) + l(\lambda b) + r(\lambda y)}} = \frac{l(a) + r(x) - l(b) - r(y)}{\sqrt{l(a) + r(x) + l(b) + r(y)}}.
\] (6)

A casual examination indicates that if \( l \) and \( r \) are the power functions of Theorem 1 then Eq. (6) and hence the conjoint Weber’s law are violated. In fact, it turns out that no reasonable psychophysical functions \( l \) and \( r \) that allow the counting model to predict the conjoint Weber’s law can be found. This fact is summarized in Theorem 2.

The proof of Theorem 2 utilizes a result of Falmagne and Iverson (1979), who showed that if the conjoint Weber’s law holds and \( l \) and \( r \) are continuous and strictly increasing, then a simple PACOME must be one of the following three types,

\[
P_{ax, by} = G(a^\theta x^\delta/b^\theta y^\delta)
\] (7)

\[
P_{ax, by} = Q(a/x, b/y)
\] (8)

\[
P_{ax, by} = G\left[\frac{a^\theta + \delta x^\delta}{b^\theta + \delta y^\delta}\right]
\] (9)

where \( G \) is continuous and strictly increasing and \( Q \) is continuous and strictly increasing in the first argument and strictly decreasing in the second. The proof consists of showing that Eq. (5) cannot be written in any of these forms.

**Theorem 2.** If the subject employs a counting strategy and if \( l \) and \( r \) are continuous and strictly increasing and there exist constants \( a_0 \) and \( x_0 \) such that \( l(a_0) = r(x_0) = 0 \), then the conjoint Weber’s law fails.

**Proof.** First note that if \( y = x \) then Eq. (7) becomes

\[
P_{ax, bx} = G(a^\theta/b^\theta)
\]

which does not depend on \( x \). However, we already noted that if \( y = x \) and \( a > b \), then Eq. (5) decreases in \( x \). Therefore Eq. (5) cannot be written in the form of Eq. (7).

Second, if \( b = y \) then Eq. (8) becomes

\[
P_{ax, yy} = Q(a/x, 1)
\]

where \( x > 0, a \) and \( b \) are both positive.
which does not depend on \( b \) or \( y \). On the other hand, if \( b = y \) then Eq. (5) becomes

\[
P_{ax,by} = \Phi \left[ \frac{\sqrt{n \Delta t} \left( l(a) + r(x) - l(y) - r(y) \right)}{\sqrt{l(a) + r(x) + l(y) + r(y)}} \right]
\]

which depends on \( y \) even if \( l = r \). Therefore, Eq. (5) cannot be written in the form of Eq. (8).

A different approach is needed to show that Eq. (5) cannot be written in the form of Eq. (9). Falmagne and Iverson (1979) show that if the simple PACOME takes the form of Eq. (9), then \( l \) and \( r \) must take the form

\[
l(a) = A_1 a^\beta + B_1
\]

and

\[
r(x) = A_2 x^\beta + B_2,
\]

where \( A_1, A_2, B_1, B_2, \) and \( \beta \) are some constants. Substituting these into Eq. (6), cross-multiplying, squaring both sides, and simplifying indicate that the conjoint Weber’s law holds if

\[
\lambda^\beta (A_1 a^\beta + A_2 x^\beta + A_1 b^\beta + A_2 y^\beta) + 2B_1 + 2B_2 = \lambda^{2\beta} (A_1 a^\beta + A_2 x^\beta + A_1 b^\beta + A_2 y^\beta) + \lambda^{2\beta} (2B_1 + 2B_2).
\]

Solving this equation for \( \lambda^\beta \) yields two solutions: either

\[
\lambda^\beta = 1
\]

or

\[
\lambda^\beta = \frac{-2(B_1 + B_2)}{A_1 a^\beta + A_2 x^\beta + A_1 b^\beta + A_2 y^\beta + 2B_1 + 2B_2}.
\]

In the first solution, the conjoint Weber’s law holds only in the trivial \( \lambda = 1 \) case and in the second solution, \( \lambda \) depends on specific values of \( a, b, x, \) and \( y \) and thus the conjoint Weber’s law does not hold in general. Therefore, Eq. (5) cannot be written in the form of Eq. (9).

Empirical support of the conjoint Weber’s law therefore rules out a counting strategy. If the psychophysical functions are the power functions described in Theorem 1 and \( a \geq b \) and \( x > y \), then the counting model predicts

\[
P_{(ia)(ix), (ib)(iy)} = \Phi \left[ \frac{\lambda^{\beta/2} \sqrt{n \Delta t} \left( x a^\beta + \gamma x^\beta - \alpha b^\beta - \gamma y^\beta \right)}{\sqrt{x a^\beta + \gamma x^\beta + \alpha b^\beta + \gamma y^\beta}} \right]
\]

\[
\geq \Phi \left[ \frac{\sqrt{n \Delta t} \left( x a^\beta + \gamma x^\beta - \alpha b^\beta - \gamma y^\beta \right)}{\sqrt{x a^\beta + \gamma x^\beta + \alpha b^\beta + \gamma y^\beta}} \right] = P_{ax,by}
\]
for \( \lambda > 1 \), or in other words that the conjoint Weber's inequality holds. Thus whereas the timing model naturally predicts the conjoint Weber's law, the counting model naturally predicts the conjoint Weber's inequality.

Recall that empirically, the conjoint Weber's inequality is a reasonable prediction. It seems to hold when the conjoint Weber's law fails (Falmagne & Iverson, 1979). Green and Luce (1973) report empirical evidence that human subjects may be able to switch back and forth between counting and timing strategies. This suggests the possibility that subjects might be using a timing strategy in experiments where the conjoint Weber's law holds and a counting strategy in experiments where the conjoint Weber's inequality holds.

It therefore appears that in addition to their importance to the measurement theoretic approach to psychophysics, conjoint Weber laws have great potential to differentiate between alternative processing models of psychophysics.

**The Single Channel Weber's Law**

With monaural stimuli, the conjoint Weber's law reduces to the standard single channel Weber's law

\[ P_{\lambda x, \lambda y} = P_{x, y}. \]

Luce and Green (1972) considered the relationship of Weber's law to the timing model and McGill (1967) discussed its relation to the counting model. However, these authors made specific assumptions about the form of the psychophysical function and they did not explicitly consider the ability of Weber's law to discriminate empirically between counting and timing strategies. It turns out, however, that a parallel form for each of the above results exists for the single channel Weber's law.

To see this, we can make use of the fact that Weber's law is a special case of the conjoint Weber's law in which the intensity input to one channel is always zero. To derive parallel results for Weber's law, therefore, we need only set \( h(a) \) and \( h(b) \), say, to zero in each of the above equations. Thus, the timing model predicts Weber's law to hold if and only if

\[
\frac{r(\lambda x)}{r(\lambda y)} = \frac{r(x)}{r(y)},
\]

where now \( r(x) \) is the rate of the train of neural spikes produced when stimulus \( x \) is presented. The next result follows easily from Theorem 1.

**Lemma 1.** If the subject employs a timing strategy and Weber's law holds and if \( r \) is continuous, then

\[ r(x) = ax^\beta, \]

where \( x > 0 \), \( a \) and \( \beta \) are constants, and \( a \) is nonzero.
Proof. Setting \( y = 1 \) in Eq. (10) and rearranging yields

\[
\frac{r(\lambda x)}{r(1)} = \frac{r(x)}{r(1)}
\]

which is in the same form as Eq. (4). The result follows from the Theorem 1 proof. \( \Box \)

From the development that led to Eq. (6), it is clear that the counting model predicts Weber’s law to hold if and only if

\[
\frac{r(\lambda x) - r(\lambda y)}{\sqrt{r(\lambda x) + r(\lambda y)}} = \frac{r(x) - r(y)}{\sqrt{r(x) + r(y)}}.
\]  

(11)

The following result parallels Theorem 2.

THEOREM 3. If the subject employs a counting strategy and if \( r \) is continuous and strictly increasing, then Weber’s law fails.

Proof. Setting \( x = 0 \) reduces Eq. (11) to

\[
\frac{r(0) - r(\lambda y)}{\sqrt{r(0) + r(\lambda y)}} = \frac{r(0) - r(y)}{\sqrt{r(0) + r(y)}}.
\]  

(12)

Squaring both sides, cross-multiplying, and simplifying lead to

\[
r(\lambda y) = \frac{r(0) [3r(0) - r(y)]}{r(0) + r(y)}.
\]  

(13)

If \( r(y) \) is strictly increasing, then \( r(0) + r(y) \) is increasing but \( 3r(0) - r(y) \) is decreasing. Therefore the right hand side of (13) is decreasing in \( y \). But if \( r(y) \) is increasing, then \( r(\lambda y) \) is increasing for positive \( \lambda \). Therefore (13) cannot hold for all values of \( \lambda \) and \( y \) and therefore neither can (11). \( \Box \)

Finally, note that, as with the conjoint Weber’s law, if the psychophysical functions are power functions then the counting model predicts the Weber’s inequality

\[
P_{\lambda x, \lambda y} \geq P_{x, y}
\]

for \( \lambda \geq 1 \). Thus the single channel results perfectly parallel the conjoint results. The timing model naturally predicts Weber’s law and the counting model naturally predicts Weber’s inequality.

1 Note that Theorem 3 assumes that \( x = 0 \) is in the domain of stimuli. Actually, an anonymous reviewer pointed out that it is only necessary that 0 be a cluster point in the domain of stimuli. Let the set of all stimuli be \( X = \{x_1, x_2, \ldots \} \). If 0 is a cluster point of \( X \), then there exists a sequence \( \langle x_n \rangle \) converging to 0. Further, if \( r \) is continuous, then \( \langle r(x_n) \rangle \) converges to \( r(0) \). Therefore, if (11) holds for all \( x \), then

\[
\lim_{n \to \infty} \frac{r(\lambda x_n) - r(\lambda y)}{\sqrt{r(\lambda x_n) + r(\lambda y)}} = \lim_{n \to \infty} \frac{r(x_n) - r(y)}{\sqrt{r(x_n) + r(y)}}
\]

which implies Eq. (12).
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